UDC 534

EXISTENCE OF STATIONARY SOLUTIONS OF THE NONLINEAR WAVE EQUATION

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The existence of a denumerable set of stationary solutions with cylindrical or spherical symmetry for a nonlinear wave equation is proved directly by the variational method.

Stationary solutions of the nonlinear wave equation make it possible to estimate the characteristic space and energy parameters of light beams in a nonlinear medium. They were the subject of detailed investigation of the cubic wave equation [1–4]. At considerable intensity of light waves the dependence of permittivity $\varepsilon = \varepsilon(|E|)$ ceases to be quadratic and becomes a complex function of field amplitude $|E|$. The determination of the exact functional dependence $\varepsilon(|E|)$ for considerable fields in the presence of the Kerr orientational effect is very difficult and can be completed only in the case of the simplest model of noninteracting molecules [5]. It was established with the use of this model that $\varepsilon(|E|)$ is a monotonically increasing bounded function for all values of the amplitude of $|E|$. An extension of the theory to interacting molecules does not alter the monotonicity of increase and boundedness of function $\varepsilon(|E|)$. It only affects the magnitude of the field saturation and, also, the highest possible increase of the nonlinear permittivity [6, 7]. One is faced with the problem of investigation of the existence of stationary solutions without specifically determining the functional dependence $\varepsilon(|E|)$, using only its boundedness and monotonicity of increase. The method of the phase plane utilized in several papers [1, 3] cannot be applied here, since it is qualitative and has to be always checked by numerical calculations. Moreover, it should be noted that numerical calculations carried out for the simplest model of medium with saturating nonlinearity [8, 9] in no way indicate the existence of stationary solutions for real media with the Kerr orientation effect.

Using the direct variational method [10, 11] it is shown here that the monotonicity of increase and the boundedness of function $\varepsilon(|E|)$ are sufficient conditions of the existence of a denumerable set of stationary solutions with cylindrical or spherical symmetry possessing finite energy. The obtained results are applicable not only in nonlinear optics, but also in the nonlinear field theory of elementary particles, where, owing to the instability of solutions of the cubic wave equation, it is necessary to introduce in the equation more complex nonlinear terms [8, 12].

1. Existence of the fundamental mode. We call stationary the solution of equation

$$i\frac{\partial E}{\partial \tau} + \Delta E + f(|E|^2)E = 0 \qquad (1.1)$$

a solution of the form $E = \varphi(r)e^{j\gamma\tau}$, where $\varphi(r)$ is a function with an integrable square which satisfies the following equation and boundary conditions:

$$\frac{d^2\varphi}{dr^2} + \frac{m}{r}\frac{d\varphi}{dr} - \gamma\varphi + f(\varphi^2)\varphi = 0 \qquad (\gamma > 0) \qquad (1.2)$$

$$\left.\frac{d\varphi}{dr}\right|_{r=0} = 0, \qquad \varphi(\infty) = 0 \tag{1.3}$$

where $m = 0,1,2$ is determined by the dimension of the Laplacian in Eq. (1.1).

To prove the existence of solutions we shall consider the following variational problem: to find the maximum of functional

$$G(\varphi) = \int F(\varphi^2)\, dv, \quad F(\varphi^2) = \int_0^{\varphi^2} f(\eta)\, d\eta, \quad dv = r^m dr$$

for the class of positive functions with piecewise smooth derivatives which satisfy boundary conditions (1.3) and the normalization

$$N(\varphi) = \int [(\nabla\varphi)^2 + \gamma\varphi^2]\, dv = R = \text{const} \tag{1.4}$$

Function $f(\eta)$ is positive and bounded for all η, hence functional $G(\varphi)$ for the class of introduced functions is bounded above

$$G(\varphi) \leqslant \int f\varphi^2 dv \leqslant \max f(\varphi^2)\, R/\gamma$$

Use is made here of the relationship

$$F(\varphi^2) = f\varphi^2 - \int_0^{\varphi^2} f'_\eta \eta\, d\eta \leqslant f\varphi^2$$

which is valid for $f_\eta' > 0$.

In what follows we assume that $f(0) = 0$, $\lim \eta \to \infty\ f = M$ and $f_\eta' > 0$ for all $\eta > 0$. Owing to the upper boundedness the functional $G(\varphi)$ has an exact upper boundary λ and there exists a maximizing succession of functions $y_n > 0$ which satisfy the specified conditions, and for $\lim\ n \to \infty\ G(y_n) = \lambda$.

Let us prove that the maximizing succession y_n can be always chosen so that its limit function y_0 is a positive solution of the equation

$$\frac{d^2 y_0}{dr^2} + \frac{m}{r}\frac{dy_0}{dr} - \gamma y_0 + \alpha(R) f(y_0^2)\, y_0 = 0 \tag{1.5}$$

where $\alpha(R)$ is a continuous function of the normalization constant R. We juxtapose to each function y_n the corresponding function u_n which satisfies the equation

$$\frac{d^2 u_n}{dr^2} + \frac{m}{r}\frac{du_n}{dr} - \gamma u_n + \alpha_n f(y_n^2)\, y_n = 0 \tag{1.6}$$

and boundary conditions (1.3). In this equation α_n is determined by the normalization condition $N(u_n) = R$. The solution of Eq. (1.6) may be written as

$$u_n = \alpha_n \int_0^\infty f(y_n^2)\, y_n g_m(r, \xi)\, d\xi \tag{1.7}$$

where $g_m(r, \xi)$ is Green's function of the homogeneous equation (1.6) with boundary conditions (1.3). The properties of Green's function $g_m(r, \xi)$, boundedness of $f(\varphi^2)$, and the normalization y_n imply the homogeneous boundedness of succession u_n

$$0 < u_n(r) < \alpha_n \left(\int_0^\infty f^2 y_n^2 \xi^m d\xi\right)^{1/2} \left(\int_0^\infty \frac{g_m^2(r,\xi)}{\xi^m}\, d\xi\right)^{1/2} < cR \tag{1.8}$$

From (1.8) and (1.4) we obtain
$$\int_0^\infty \left(\frac{du_n}{dr}\right)^2 dr < c_1$$
which implies the equicontinuity of succession u_n [13].

Let us prove that for $\lim\ n \to \infty\ G(u_n) = \lambda$. Since functions u_n satisfy the same conditions as y_n, hence $G(u_n) \leqslant \lambda$ for all n. By virtue of condition that $f_\eta' > 0$ the inequality
$$G(u_n) \geqslant G(y_n) + \int (u_n^2 - y_n^2) f(y_n^2)\, dv \tag{1.9}$$
is valid. Multiplying (1.6) by u_n, integrating over the whole space, and using Buniakowski's inequality, we obtain
$$R^2 \leqslant \alpha_n^2 \int f(y_n^2)\, y_n^2\, dv \int f(y_n^2)\, u_n^2\, dv \tag{1.10}$$
Multiplying (1.6) by y_n, integrating, and using again Buniakowski's inequality, after some simple transformations we obtain
$$R \geqslant \alpha_n \int f(y_n^2)\, y_n^2\, dv \tag{1.11}$$
It follows from (1.9)–(1.11) that $\lambda \geqslant G(u_n) \geqslant G(y_n)$ and $\lim G(u_n) = \lim G(y_n) = \lambda$ for $n \to \infty$.

We have thus proved that the succession u_n is a maximizing one uniformly bounded and equicontinuous. It is therefore always possible to select as the input maximizing succession y_n one which is uniformly bounded and equicontinuous. According to a known theorem it is possible to chose from among such successions ones that converge to continuous limit functions y_0 and u_0, with $G(y_0) = G(u_0) = \lambda$ [13]. The inequalities (1.9)–(1.11) become equalities for the limit functions only if condition
$$\int (y_0 - u_0)^2\, dv = 0 \tag{1.12}$$
is satisfied. It follows from (1.12) and the continuity of u_0 and y_0 that at every point $y_0(r) = u_0(r)$. Since (1.8) implies that y_n and u_n are bounded functions and
$$\int_t^\infty f(y_n^2)\, y_n g_m(r, \xi)\, d\xi < cMR \int_0^\infty g_m(r, \xi)\, d\xi < \infty$$
hence it is possible to pass to the limit of the integrand in (1.7)
$$y_0 = \alpha \int_0^\infty f(y_0^2)\, y_0 g_m(r, \xi)\, d\xi, \qquad \alpha = \lim_{n\to\infty} \alpha_n \tag{1.13}$$
Differentiation of (1.13) with respect to r shows that y_0 satisfies Eq. (1.5) and boundary conditions (1.3). Let us prove that for $0 < \gamma < M$ it is possible to chose R so that $\alpha(R) = 1$. From (1.5) we have the upper and lower estimates for α
$$\frac{R}{G(y_0)} > \alpha = \frac{R}{\int f y_0^2 dv} > \frac{\gamma}{\max f(y_0^2)}$$
It is evident from (1.8) that
$$\lim (\max y_0^2) = \lim (\max f(y_0^2)) = 0, \qquad R \to 0$$
Hence $\alpha(R) \to \infty$ when $R \to 0$. Let us now prove that $\alpha(R) < 1$ when $R > R^*$. For this it is sufficient to find a positive function Ψ for which the inequality
$$N(\Psi) / G(\Psi) = R / G(\Psi) < 1$$

is satisfied for the normalization $N(\Psi) = R > R^*$.

The positive solution of problem

$$\frac{d^2\Psi}{dr^2} + \frac{m}{r}\frac{d\Psi}{dr} - \gamma\Psi + M'\Psi = 0 \quad (\gamma < M' < M) \tag{1.14}$$

$$\left.\frac{d\Psi}{dr}\right|_{r=0} = 0, \quad \Psi(r_1) = 0, \quad \Psi(r) = 0, \quad r > r_1$$

is such a function. Here $r_1 = \pi / \sqrt{M' - \gamma}$ for $m = 0, 2$, $r_1 = x_1 / \sqrt{M' - \gamma}$ for $m = 1$ and $x_1 = 2.41$ is the smallest root of the zero-order Bessel function. It is possible to prove that for $M' < M = \max f$ and $f'_\eta > 0$ there exists such number $\beta_1 > 0$ that the inequality

$$M' \int \beta^2\Psi^2 dv < \int F(\beta^2\Psi^2)\, dv = G(\beta\Psi) \tag{1.15}$$

is satisfied for $\beta > \beta_1$. It follows from (1.14) and (1.15) that $N(\beta\Psi) / G(\beta\Psi) < 1$ when $\beta > \beta_1$. If we select the normalization constant $R > R^* = N(\beta_1\Psi)$ then

$$\alpha(R) < \frac{R}{\max G} < \frac{R}{G(\beta\Psi)} < 1$$

Thus $\alpha(R) \to \infty$ when $R \to 0$ and $\alpha(R) < 1$ when $R > R^*$, hence there exists a normalization constant $R^* > R_0 > 0$ such that $\alpha(R_0) = 1$ and the limit function of the considered variational problem is a positive solution of Eq. (1.2). For $\gamma > M$ Eq. (1.2) has no solutions with an integrable square. Multiplying (1.2) by φ and integrating, we obtain

$$M \int \varphi^2 dv \geqslant \int f\varphi^2 dv \geqslant \gamma \int \varphi^2 dv + k \left(\int \varphi^2 dv\right)^2 \Big/ \int r^2\varphi^2 dv. \tag{1.16}$$

We have used here the inequality

$$\int (\nabla\varphi)^2\, dv \int r^2\varphi^2 dv \geqslant k(m) \left(\int \varphi^2 dv\right)^2$$

where $k(m)$ is a constant which depends only on the dimension of space m [14]. It follows from (1.16) that

$$M > \gamma \qquad \langle r^2 \rangle = \frac{\int r^2\varphi^2 dv}{\int \varphi^2 dv} \geqslant \frac{k}{M - \gamma} \tag{1.17}$$

In a medium with condensing nonlinearity the smallest dimension of a stationary solution is bounded below by $k / (M - \gamma)$. If function (φ^2) is unbounded, the effective dimension of stationary distribution $\langle r^2 \rangle$ can be as small as desired.

Let us briefly consider the basic properties of solutions of Eq. (1.2) in the case of a finite region. The proof of existence of a positive solution of that equation with boundary conditions $\varphi(r_1) = \varphi(\infty) = 0$ and $r_1 \leqslant r < \infty$ for $m = 1,2$ is similar. It can be proved that for $r_1 \to \infty$ the functional

$$H(\varphi) = \int \left[\left(\frac{d\varphi}{dr}\right)^2 + \gamma\varphi^2 - F(\varphi^2)\right] dv \to \infty \tag{1.18}$$

For $m = 0$ (one-dimensional space) this boundary problem has no solution. Multiplying (1.2) in this case by $d\varphi / dr$, we obtain the first integral

$$\left(\frac{d\varphi}{dr}\right)^2 + \gamma\varphi^2 - F(\varphi^2) = c \tag{1.19}$$

Since for $r \to \infty$ we have $d\varphi / dr \to 0$ and $\varphi \to 0$, hence for a stationary solution $c = 0$. This implies that at point $r = r_1$ not only $\varphi(r_1) = 0$, but also $d\varphi / dr_1 = 0$. Equation (1.19) with boundary conditions $d\varphi/dr_1 = \varphi(r_1) = \varphi(\infty) = 0$ has a unique solution $\varphi \equiv 0$. This shows that in the one-dimensional case only the fundamental mode exists, while any higher modes are absent.

For boundary conditions $d\varphi / dr\,|_{r=0} = \varphi(r_1) = 0$ or $\varphi(r_1) = \varphi(r_2) = 0$ the proof of existence of positive solutions in a finite region of fairly large size is similar to that in the case of an infinite region. As implied by inequality (1.17), Eq. (1.2) has no finite solutions when the size of the region is small $(r_1^2 < k / (M - \gamma))$. It can be proved that the exact minimum size of a region for which Eq. (1.2) has no solution is equal to the minimum size of a region in which for $M' = M$ and related boundary conditions there exist finite positive solutions of Eq. (1.4). For boundary conditions $d\varphi / dr\,|_{r=0} = \varphi(r_1) = 0$ the critical size of the region is $r_* = \pi / \sqrt{M - \gamma}$ when $m = 0, 2$ and $r_* = x_1 / \sqrt{M - \gamma}$ when $m = 1$. When the region approaches its critical size $\max\varphi$ and the derivatives at the region boundary infinitely increase

$$\max\varphi \to \infty, \quad \left|\frac{d\varphi}{dr_1}\right| \to \infty, \quad \left|\frac{d\varphi}{dr_2}\right| \to \infty \tag{1.20}$$

In a region of smaller than critical size Eq. (1.2) has no finite solutions.

2. Higher modes. When function $f(\varphi^2)$ is unbounded, Eq. (1.2) for $m = 1,2$ can have in addition to the positive solution (fundamental mode) a denumerable set of solutions with finite energy which vanish exactly n times, where n is the number of the mode [1-4, 11]. A similar theorem is valid for a bounded positive monotonically increasing function $f(\varphi^2)$.

Let us divide the whole interval $0 \leqslant r < \infty$ into n parts so that in each subinterval $r_i \leqslant r \leqslant r_{i+1}$, $i = 1,2,\ldots, n$, Eq. (1.2) would have a positive solution φ_i which satisfies the boundary condition $\varphi_i(r_i) = \varphi_i(r_{i+1}) = 0$, $i = 2,3,\ldots, n$, (for the first interval it must satisfy the boundary condition $d\varphi_1/dr|_{r=0} = 0$ and $\varphi_1(r_2) = 0$). It follows from the results obtained in Sect. 1 that for $m = 1,2$ such subdivision is always possible. We introduce a continuous function $\Psi_{n-1}(r)$ which in each subinterval $[r_i, r_{i+1}]$ is equal to the solution $(-1)^{i-1}\varphi_i(r)$. Evidently $\Psi_{n-1}(r)$ satisfies Eq. (1.2) throughout the space except, possibly, at points $r = r_i$, $i = 2,3,\ldots,n$, where its derivatives may be discontinuous.

Let us prove that the division into subspaces can be made so that at all points $r \geqslant 0$ the function $\Psi_{n-1}(r)$ would have continuous first and second derivatives and would be a solution of Eq. (1.2) throughout the space. For this it is sufficient to chose a subdivision which would yield the minimum of functional [11]

$$H_{n-1}(r_2, r_3, \ldots, r_n) = \int \left[\left(\frac{d\Psi_{n-1}}{dr}\right)^2 + \gamma\Psi_{n-1}^2 - F(\Psi_{n-1}^2)\right] dv =$$
$$\sum_{i=1}^{n} \int_{r_i}^{r_{i+1}} \left[\left(\frac{d\varphi_i}{dr}\right)^2 + \gamma\varphi_i^2 - F(\varphi_i^2)\right] dv = \sum_{i=1}^{n} \int_{r_i}^{r_{i+1}} [f\varphi_i^2 - F(\varphi_i^2)]\, dv \geqslant 0$$

which is a continuous function of the variables $r_2, r_3, \ldots, r_n$ [15]. The existence of minimum follows from the positiveness of functional H_{n-1} and from the properties of positive solutions of (1.18) and (1.19). According to [15]

$$\frac{\partial H_{n-1}}{\partial r_i} = \left[\left(\frac{d\varphi_{i-1}}{dr}\right)^2 - \left(\frac{d\varphi_i}{dr}\right)^2\right] r^m \Big|_{r=r_i}, \quad i = 2, 3, \ldots, n \tag{2.1}$$

Since at the minimum all derivatives $\partial H_{n-1} / \partial r_i = 0$, hence it follows from (2.1) that $d\Psi_{n-1} / dr$ is continuous for all $r \geqslant 0$. Consequently there exists a division into subspaces for which function $\Psi_{n-1}(r)$ has continuous derivatives, is a solution of Eq. (1.2), and has exactly $n-1$ zeros.

On the basis of obtained results it is possible to state that the monotonic increase, boundedness, and positiveness of functions $f(\varphi^2)$ are sufficient conditions for the existence of a denumerable set of stationary solutions with cylindrical or spherical symmetry.

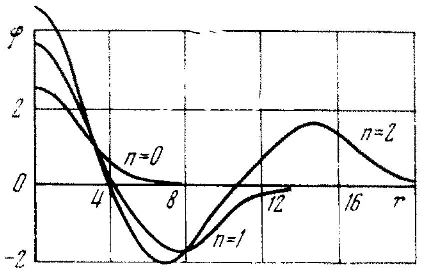

Fig. 1

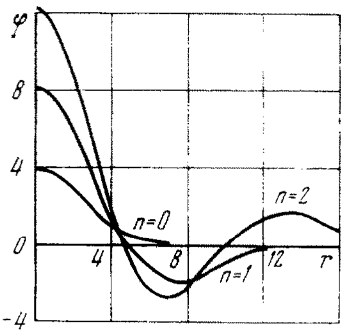

Fig. 2

The expounded theory was tested by numerical solution of Eq. (1.2) in two- and three-dimensional cases with $f = \varphi^2 / (1 + \varphi^2)$. A denumerable set of symmetric solutions with finite energy was obtained in complete agreement with the theory. The related distribution of amplitudes for the first three modes for $\gamma = 0.5$ are shown in Fig. 1 ($m = 1$) and Fig. 2 ($m = 2$).

An extension of this theorem to positive monotonically increasing unbounded functions $f(\varphi^2)$ of power increase is possible. The existence of solutions of the related variational problem is proved with the use of interpolation inequalities. All subsequent analysis is effected by following the scheme proposed here. In particular, a denumerable set of axially symmetric solutions exists for functions $f(\varphi^2)$ of the form

$$f(\varphi^2) = \sum_{k=1}^{N} c_k \varphi^{2k}, \quad f \geqslant 0, \quad \frac{df}{d\varphi^2} \geqslant 0, \quad N < \infty$$

Note that unbounded functions $f(\varphi^2)$ have solutions in any arbitrarily small region. If function $f(\varphi^2)$ does not satisfy the conditions of theorems presented above, the existence of stationary solutions is determined by the particular form of function $f(\varphi^2)$ and requires special investigations in each case [8, 9].

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